

ON A GEOMETRIC REALIZATION OF C^* -ALGEBRAS

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ABSTRACT. Further to the functional representations of C^* -algebras proposed in [1], we consider in this article the uniform Kähler bundle (in short, UKB) description of some C^* -algebraic subjects. In particular, we obtain an one-to-one correspondence between closed ideals of a C^* -algebra \mathcal{A} and full uniform Kähler subbundles over open subsets of the base space of the UKB associated with \mathcal{A} . In addition, we will present a geometric description of the pure state space of hereditary C^* -subalgebras and show that if \mathcal{B} is a hereditary C^* -subalgebra of \mathcal{A} , the UKB of \mathcal{B} is a kind of Kähler subbundle of the UKB of \mathcal{A} . As a simple example, we consider hereditary C^* -subalgebras of the C^* -algebra of compact operators on a Hilbert space. Finally, we remark that hereditary C^* -subalgebras also naturally can be characterized as uniform holomorphic Hilbert subbundles.

1. INTRODUCTION AND PRELIMINARIES

For every commutative unital C^* -algebra \mathcal{A} , it is well known that \mathcal{A} can be faithfully represented as the algebra of the continuous functions on a compact Hausdorff space. More precisely, Gelfand showed that \mathcal{A} can be viewed as the space of continuous functions on the pure state space of \mathcal{A} , equipped with the w^* -topology. Gelfand's construction sets up an isomorphism between the categories of commutative C^* -algebras (with unit) and compact Hausdorff spaces. By means of this duality, any statement about commutative C^* -algebras can be rephrased in the language of topological spaces, and vice versa.

It is rather natural to attempt a non-commutative version of this construction and try to understand C^* -algebras in terms of continuous functions on certain spaces. A way to obtain such a concrete realization had been proposed in [1]. The space that considered there is still the set of pure states as in the commutative case, equipped with more complicated topological and geometric structures. More precisely, the set \mathcal{P} will be viewed as a topological bundle of infinite-dimensional Kähler manifolds over the base space of the spectrum of the C^* -algebra (i.e. the topological space of unitary equivalence classes of non-zero irreducible representations). In fact, each fiber of an irreducible representation consists of pure states whose GNS representations are unitarily equivalent to that representation. In this paper, we use this to give a geometric realization of some C^* -algebraic subjects (such as closed ideals and hereditary C^* -subalgebras).

Firstly, we set up a one-to-one correspondence between closed ideals of a general C^* -algebra \mathcal{A} and a class of uniform Kähler subbundles of pure state space $\mathcal{P}(\mathcal{A})$ of \mathcal{A} . More

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precisely, we show that, for any subbundle $(\mathcal{E}, p_{\mathcal{A}}|_{\mathcal{E}}, \mathcal{X})$ of $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})$ such that \mathcal{X} is an open set of the spectrum $\hat{\mathcal{A}}$ of \mathcal{A} and $\mathcal{E} = p_{\mathcal{A}}^{-1}(\mathcal{X})$, there is a close ideal \mathcal{J} in \mathcal{A} with $\mathcal{X} \cong \hat{\mathcal{A}}^{\mathcal{J}}$ and $\mathcal{P}(\mathcal{J}) \cong \mathcal{E}$, and vice versa.

Secondly, we show that if \mathcal{B} is a hereditary C^* -subalgebra of a C^* -algebra \mathcal{A} , then $\mathcal{P}(\mathcal{B})$ can be viewed as a kind of Kähler subbundle of $\mathcal{P}(\mathcal{A})$ whose base space is a closed set of $\hat{\mathcal{A}}$ and every fiber can be viewed as a closed Kähler submanifold of the corresponding fiber on $\hat{\mathcal{A}}$. Moreover, we discuss a kind of geometric structure on the pure state space of a C^* -algebra and consider its relation to hereditary C^* -subalgebras. Finally, as a simple example, we give a geometric characterization of hereditary C^* -subalgebras of the C^* -algebra of compact operators on a Hilbert space.

We remark finally that, according to [3], hereditary C^* -subalgebras of a C^* -algebra \mathcal{A} also naturally can be characterized as another kind of bundles associated to \mathcal{A} , i.e., uniform holomorphic Hilbert bundles.

Before we start, let us first set some notations and recall from [1] some results concerning generalization of the Gelfand transform for non commutative unital C^* -algebras.

Notation 1.1. *All vector spaces in this article are over the complex field, unless stated otherwise. We denote by $\langle \cdot | \cdot \rangle$ the inner product on a Hilbert space \mathcal{H} (which is conjugate-linear in the first variable) and by $\mathcal{L}(\mathcal{H})$ the space of all bounded linear operators on \mathcal{H} .*

- We denote by $\mathcal{P}_{\mathcal{H}}$ the projective space of \mathcal{H} . The element of $\mathcal{P}_{\mathcal{H}}$ is denoted by $[x]$, where $x \in \mathcal{H}$ is a normalized representative of $[x]$. The unit sphere of \mathcal{H} is denoted by $\mathfrak{S}_1(\mathcal{H})$.
- We denote by $\mathcal{P}(\mathcal{A})$ the set of all pure states on a C^* -algebra \mathcal{A} endowed with w^* -topology and by $\hat{\mathcal{A}}$ the spectrum of \mathcal{A} endowed with the Jacobson topology.
- We denote by $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})$ the uniform Kähler bundle (in short, UKB) associated with a C^* -algebra \mathcal{A} , where $p_{\mathcal{A}} : \mathcal{P}(\mathcal{A}) \rightarrow \hat{\mathcal{A}}$ is the natural projection given by the GNS representation (see Section 3 in [1]).

Just as in the case of a commutative C^* -algebra, the function on the pure state space $\mathcal{P}(\mathcal{A})$ representing an element a of \mathcal{A} will be the Gelfand transform f_a of a given by

$$(1.1) \quad f_a : \mathcal{P}(\mathcal{A}) \rightarrow \mathbb{C}, \quad \omega \mapsto f_a(\omega) := \omega(a) .$$

We introduce the main results in [1] as follows.

Let $\mathcal{C}^\infty(\mathcal{P})$ be the set of all smooth complex-valued functions on a Kähler manifold \mathcal{P} . If \mathcal{P} is the total space of a UKB, then we still denote by $\mathcal{C}^\infty(\mathcal{P})$ the set of all fiberwise-smooth complex-valued functions on \mathcal{P} (see Section 3 in [1] and Section 2.2 in [5]). Note that $\mathcal{C}^\infty(\mathcal{P})$ is a $*$ -algebra with involution given by complex conjugation $f \mapsto \bar{f}$.

If \mathcal{A} is a C^* -algebra, and $\mathcal{B} = \mathcal{A} \oplus \mathbb{C}$ equipped with the canonical unital C^* -algebraic structure, then $\hat{\mathcal{B}} = \hat{\mathcal{A}} \oplus \{\pi_0\}$, $\mathcal{P}(\mathcal{B}) \cong \mathcal{P}(\mathcal{A}) \cup \{0\} \subseteq \mathcal{A}^*$ and the fibre over $\pi_0 \in \hat{\mathcal{B}}$ is $\{0\}$ (equipped with the trivial Kähler manifold structure). Moreover, for any $f \in \mathcal{C}^\infty(\mathcal{P}(\mathcal{A}))$, we extend f to a function on $\mathcal{P}(\mathcal{B})$ by setting $f(0) = 0$. Now, we have the following general form of [1, Proposition 3.2].

Theorem 1.2. ([1, Proposition 3.2]) *Let \mathcal{A} be a C^* -algebra. Then*

- (1) The Gelfand transform $a \mapsto f_a$ is a linear, involution preserving injection of \mathcal{A} into $\mathcal{C}^\infty(\mathcal{P}(\mathcal{A}))$.
- (2) The range of the Gelfand transform is the set denoted by $\mathcal{K}_u(\mathcal{P}(\mathcal{A}))$ of $f \in \mathcal{C}^\infty(\mathcal{P}(\mathcal{A}))$ such that $f, \bar{f} \star f$ and $f \star \bar{f}$ are uniformly continuous on $\mathcal{P}(\mathcal{A}) \cup \{0\}$ as well as $DDf = \bar{D}\bar{D}f = 0$, where D and \bar{D} are respectively holomorphic and anti-holomorphic parts of covariant derivative of the Kähler metric defined on each fiber of $\mathcal{P}(\mathcal{A})$, and the \star -product is as defined in [1] (see also [5, Definition 2.1]).
- (3) For any $a, b \in \mathcal{A}$, one has $f_{ab} = f_a \star f_b$ and $\|a\|^2 = \sup_{\omega \in \mathcal{P}(\mathcal{A})} (\bar{f}_a \star f_a)(\omega)$. By this norm, the $\mathcal{K}_u(\mathcal{P}(\mathcal{A}))$ is a C^* -algebra which is $*$ -isomorphic onto \mathcal{A} .

Definition 1.3. ([5, Definition 2.1 and 2.2]) Two UKBs $(\mathcal{E}, p, \mathcal{X})$ and $(\mathcal{E}', p', \mathcal{X}')$ are isomorphic if there exists a pair (ψ, ϕ) of homeomorphisms $\psi : \mathcal{E} \rightarrow \mathcal{E}'$ and $\phi : \mathcal{X} \rightarrow \mathcal{X}'$, such that $p' \circ \psi = \phi \circ p$ and any restriction $\psi|_{p^{-1}(x)} : p^{-1}(x) \rightarrow (p')^{-1}(\phi(x))$ is a holomorphic Kähler isometry for any $x \in \mathcal{X}$. We call such a pair (ψ, ϕ) a uniform Kähler isomorphism between $(\mathcal{E}, p, \mathcal{X})$ and $(\mathcal{E}', p', \mathcal{X}')$.

Corollary 1.4. ([1, Corollary 3.3]) If the UKBs $(\mathcal{P}_1, p_1, \mathcal{B}_1)$ and $(\mathcal{P}_2, p_2, \mathcal{B}_2)$ corresponding to the C^* -algebras \mathcal{A}_1 and \mathcal{A}_2 are isomorphic, then the C^* -algebras \mathcal{A}_1 and \mathcal{A}_2 are themselves $*$ -isomorphic.

By the above results, we give a geometric structure on the pure state space and obtain a correspondence between algebra and geometry as follows:

$$\begin{array}{ccc}
 \text{commutative} & \Longleftrightarrow & \text{locally compact} \\
 C^*\text{-algebra} & & \text{Hausdorff space} \\
 \cap & & \cap \\
 C^*\text{-algebra} & \Longleftrightarrow & \text{UKB associated} \\
 & & \text{with a } C^*\text{-algebra}
 \end{array}$$

2. CLOSED IDEALS, QUOTIENT ALGEBRAS AND HEREDITARY SUBALGEBRAS OF C^* -ALGEBRAS

In this section, we give a geometric characterization of quotient C^* -algebras, closed ideals and hereditary subalgebras of a C^* -algebra.

First of all, we consider quotient C^* -algebras and closed ideals of a C^* -algebra. The results are quite obvious, but we include them here for completeness and comparison.

Notation 2.1. Given a C^* -algebra \mathcal{A} , let \mathcal{S} and \mathcal{I} be a subset and a C^* -subalgebra of \mathcal{A} respectively.

- Let $\text{hull}(\mathcal{S})$ be the set of primitive ideals of \mathcal{A} containing \mathcal{S} .
- Let $\hat{\mathcal{A}}^{\mathcal{I}}$ be the set of $\pi \in \hat{\mathcal{A}}$ such that $\pi(\mathcal{I}) \neq 0$.
- Let $\mathcal{P}^{\mathcal{I}}(\mathcal{A})$ be the set of pure states of \mathcal{A} , which do not vanish on \mathcal{I} .

Definition 2.2. [4, Definition 5.1] Let $(\mathcal{E}, p, \mathcal{X})$ be a bundle, and let \mathcal{U} be a subset of \mathcal{X} . Then the restriction of $(\mathcal{E}, p, \mathcal{X})$ to \mathcal{U} , denoted by $(\mathcal{E}, p, \mathcal{X})_{\mathcal{U}}$, is the bundle $(\mathcal{E}', p', \mathcal{U})$, where $\mathcal{E}' = p^{-1}(\mathcal{U})$ and $p' = p|_{\mathcal{E}'}$.

Notation 2.3. Let \mathcal{A} be a C^* -algebra. Set

- $Bun_o(\mathcal{A}) := \{(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\mathcal{U}} \mid \mathcal{U} \subseteq \hat{\mathcal{A}} \text{ is an open subset}\}.$
- $Bun_c(\mathcal{A}) := \{(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\mathcal{V}} \mid \mathcal{V} \subseteq \hat{\mathcal{A}} \text{ is a closed subset}\}.$

Remark and Notation 2.4. Note that $\mathcal{P}_{\mathcal{H}}$ is a Kähler manifold, and the Kähler distance $d_{\mathcal{H}}$ on $\mathcal{P}_{\mathcal{H}}$ is given by $d_{\mathcal{H}}([x], [y]) = \sqrt{2} \arccos |\langle x|y \rangle|$ for any $[x], [y] \in \mathcal{P}_{\mathcal{H}}$ (see Appendix C in [1]).

Besides, for each $[\pi] \in \hat{\mathcal{A}}$, the fiber $p_{\mathcal{A}}^{-1}([\pi])$ denoted by $\mathcal{P}_{\mathcal{A}, [\pi]}$ is isomorphic, as a Kähler manifold, to the projective space of the Hilbert space \mathcal{H}_{π} . Indeed, the representation π induces a Kähler isomorphism $\Phi_{\mathcal{A}, [\pi]} : \mathcal{P}_{\mathcal{A}, [\pi]} \rightarrow \mathcal{P}_{\mathcal{H}_{\pi}}$ defined by $\Phi_{\mathcal{A}, [\pi]}(\omega) = [x_{\omega}] \in \mathcal{P}_{\mathcal{H}_{\pi}}$ for any $\omega \in \mathcal{P}_{\mathcal{A}, [\pi]}$, where x_{ω} is a canonical cyclic vector in \mathcal{H}_{π} satisfying $\omega(a) = \langle x_{\omega} | \pi(a) x_{\omega} \rangle$ (see (3.1) in [1] and Appendix D in [1]). Moreover, the Kähler distance in the $\mathcal{P}(\mathcal{A})$ can be given as follows:

$$(2.1) \quad d_{\mathcal{A}}(\omega, \omega') = \begin{cases} \sqrt{2} \arccos |\langle x_{\omega} | x_{\omega'} \rangle| & \text{if } p_{\mathcal{A}}(\omega) = p_{\mathcal{A}}(\omega'), \\ 3 & \text{otherwise,} \end{cases}$$

for any $\omega, \omega' \in \mathcal{P}(\mathcal{A})$.

Proposition 2.5. Let \mathcal{A} be a C^* -algebra. For any closed ideal \mathcal{J} of \mathcal{A} , one has:

- The UKB $(\mathcal{P}(\mathcal{J}), p_{\mathcal{J}}, \hat{\mathcal{J}})$ with respect to \mathcal{J} is uniformly Kähler isomorphic to $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\hat{\mathcal{A}}^{\mathcal{J}}}$.
- The quotient C^* -algebra \mathcal{A}/\mathcal{J} 's UKB $(\mathcal{P}(\mathcal{A}/\mathcal{J}), p_{\mathcal{A}/\mathcal{J}}, \hat{\mathcal{A}}/\mathcal{J})$ is uniformly Kähler isomorphic to $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}}$.

Proof: We define by $f : \rho \mapsto \rho|_{\mathcal{J}}$ the map from $\mathcal{P}^{\mathcal{J}}(\mathcal{A})$ to $\mathcal{P}(\mathcal{J})$, and by $g : \pi \mapsto \pi|_{\mathcal{J}}$ the map from $\hat{\mathcal{A}}^{\mathcal{J}}$ to $\hat{\mathcal{J}}$. It is well known that f and g are both homeomorphisms satisfying $g \circ p_{\mathcal{A}} = p_{\mathcal{J}} \circ f$ (see [2, Section 2.11 and 3.2]). For any $[\pi] \in \hat{\mathcal{A}}^{\mathcal{J}}$ and $\rho \in \mathcal{P}_{\mathcal{A}, [\pi]}$, we have $\mathcal{H}_{\pi|_{\mathcal{J}}} = \mathcal{H}_{\pi}$ and $x_{\rho} = x_{\rho|_{\mathcal{J}}}$. So $\Phi_{\mathcal{A}, [\pi]} \circ f \circ \Phi_{\mathcal{A}, [\pi]}^{-1}$ is an identity map on $\mathcal{P}_{\mathcal{H}_{\pi}}$, namely, f is a holomorphic Kähler isometry. As a result, (f, g) is a uniform Kähler isomorphism between $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\hat{\mathcal{A}}^{\mathcal{J}}}$ and $(\mathcal{P}(\mathcal{J}), p_{\mathcal{J}}, \hat{\mathcal{J}})$. Hence (a) holds.

To prove (b), we denote by h the quotient map from \mathcal{A} to \mathcal{A}/\mathcal{J} . It is well known that we can define by $f' : \rho \mapsto \rho \circ h$ the homeomorphism from $\mathcal{P}(\mathcal{A}/\mathcal{J})$ to $\mathcal{P}(\mathcal{A}) \setminus \mathcal{P}^{\mathcal{J}}(\mathcal{A})$ and by $g' : \pi \mapsto \pi \circ h$ the homeomorphism from $\hat{\mathcal{A}}/\mathcal{J}$ to $\hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}$ (see [2, Section 2.11 and 3.2]). Similar to the proof of (a), we can prove that (f', g') is a uniform Kähler isomorphism between $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}}$ and $(\mathcal{P}(\mathcal{A}/\mathcal{J}), p_{\mathcal{A}/\mathcal{J}}, \hat{\mathcal{A}}/\mathcal{J})$. \square

Proposition 2.6. Let \mathcal{A} be a C^* -algebra. Then

- The correspondence $\mathcal{J} \mapsto (\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\hat{\mathcal{A}}^{\mathcal{J}}}$ is a bijection from the set of closed ideals of \mathcal{A} onto $Bun_o(\mathcal{A})$, and $\mathcal{J} = \ker \bigoplus_{[\pi] \in \hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}} \pi$.
- The correspondence $\mathcal{A}/\mathcal{J} \mapsto (\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}}$ is a bijection from the set of the quotient C^* -algebras of \mathcal{A} onto $Bun_c(\mathcal{A})$.

Proof: It follows from the fact that the correspondence $\mathcal{J} \mapsto \hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}$ is a bijection from the closed ideals of \mathcal{A} to the closed subsets of $\hat{\mathcal{A}}$. Moreover, we have $\mathcal{J} = \bigcap_{[\pi] \in \hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}} \ker \pi = \ker \bigoplus_{[\pi] \in \hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{J}}} \pi$. \square

In the rest of this section, we consider hereditary C^* -subalgebras.

Let \mathcal{B} be a nonzero hereditary C^* -subalgebra of a C^* -algebra \mathcal{A} . There is an injective map Ξ from $\mathcal{P}(\mathcal{B})$ to $\mathcal{P}(\mathcal{A})$ such that the image of τ in $\mathcal{P}(\mathcal{B})$ under Ξ is the unique extension of τ to \mathcal{A} . Moreover, every element in $\hat{\mathcal{B}}$ has the form $[\pi_{\mathcal{B}}] = [(\pi|_{\mathcal{B}}, \pi(\mathcal{B})\mathcal{H}_{\pi})]$ for a unique $\pi \in \hat{\mathcal{A}}^{\mathcal{B}}$.

First, we discuss the relation between $\Xi(\mathcal{P}(\mathcal{B}))$ and $\mathcal{P}^{\mathcal{B}}(\mathcal{A})$.

Proposition 2.7. *Let \mathcal{B} be a nonzero hereditary C^* -subalgebra of a C^* -algebra \mathcal{A} and $[\pi] \in \hat{\mathcal{A}}^{\mathcal{B}}$. Then*

(a) *if $\pi(\mathcal{B})\mathcal{H}_{\pi} = \mathcal{H}_{\pi}$, there is a unique bijection $\Theta : \mathcal{P}_{\mathcal{A},[\pi]} \rightarrow \mathcal{P}_{\mathcal{B},[\pi_{\mathcal{B}}]}$ such that $\Theta(\rho) = \rho|_{\mathcal{B}}$ and $\Theta = \Xi^{-1}$.*

(b) *if $\pi(\mathcal{B})\mathcal{H}_{\pi} \subsetneq \mathcal{H}_{\pi}$, there is a unique surjection $\Theta : \mathcal{P}_{\mathcal{A},[\pi]} \cap \mathcal{P}^{\mathcal{B}}(\mathcal{A}) \rightarrow (0, 1] \times \mathcal{P}_{\mathcal{B},[\pi_{\mathcal{B}}]}$ such that $\rho|_{\mathcal{B}} = t\rho'$ if $\Theta(\rho) = (t, \rho')$.*

Proof: (a) If $\pi(\mathcal{B})\mathcal{H}_{\pi} = \mathcal{H}_{\pi}$, then $\mathcal{H}_{\pi_{\mathcal{B}}} = \mathcal{H}_{\pi}$. So, for any $\rho \in \mathcal{P}_{\mathcal{A},[\pi]}$, we have $\rho|_{\mathcal{B}} \in \mathcal{P}_{\mathcal{B},[\pi_{\mathcal{B}}]}$ and $\Xi(\Theta(\rho)) = \rho$. On the other hand, for each $\rho' \in \mathcal{P}_{\mathcal{B},[\pi_{\mathcal{B}}]}$, we have $\Theta(\Xi(\rho')) = \rho'$. Hence (a) holds.

(b) We first show that Θ is well-defined. For any $\rho \in \mathcal{P}_{\mathcal{A},[\pi]} \cap \mathcal{P}^{\mathcal{B}}(\mathcal{A})$, by [6, Corollary 5.5.3], there exists a pair $(t, \rho') \in (0, 1] \times \mathcal{P}(\mathcal{B})$ such that $\rho|_{\mathcal{B}} = t\rho'$. If there is another pair $(s, \omega) \in (0, 1] \times \mathcal{P}_{\mathcal{B},[\pi_{\mathcal{B}}]}$ such that $\rho|_{\mathcal{B}} = s\omega$, then $t = s$ and $\rho' = \omega = \frac{1}{t}\rho|_{\mathcal{B}}$ as $\|t\rho'\| = \|s\omega\|$ as well as $\|\rho'\| = \|\omega\| = 1$. Next, we show that Θ is a surjection. For any $(t, \rho') \in (0, 1] \times \mathcal{P}_{\mathcal{B},[\pi_{\mathcal{B}}]}$, let $(\mathcal{H}_{\rho'}, \pi_{\rho'})$ be the GNS representation of \mathcal{B} associated with ρ' . Then $\pi_{\mathcal{B}}$ is unitarily equivalent to $(\mathcal{H}_{\rho'}, \pi_{\rho'})$. Let $x \in \mathfrak{S}_1(\pi(\mathcal{B})\mathcal{H}_{\pi})$ be the canonical cyclic vector with $\rho'(b) = \langle x | \pi_{\mathcal{B}}(b)x \rangle$. Set $y = \sqrt{t}x \in \pi(\mathcal{B})\mathcal{H}_{\pi}$, so that $\|y\| = \sqrt{t}$. Since $\pi(\mathcal{B})\mathcal{H}_{\pi} \subsetneq \mathcal{H}_{\pi}$, we can choose a $z \in (\pi(\mathcal{B})\mathcal{H}_{\pi})^{\perp}$ satisfying $\|z\| = \sqrt{1-t}$. Let P be a projection $\mathcal{H}_{\pi} \mapsto \pi(\mathcal{B})\mathcal{H}_{\pi}$ and put $h = y + z \in \mathcal{H}_{\pi}$. It's clear that $\|h\| = 1$ and $P(h) = y$. For any $v, w \in \mathcal{H}_{\pi}$ and $b \in \mathcal{B}$,

$$\begin{aligned} \langle w | \pi(b)P(v) \rangle &= \langle \pi(b^*)w | P(v) \rangle = \langle P\pi(b^*)w | v \rangle = \langle \pi(b^*)w | v \rangle \\ &= \langle w | \pi(b)(v) \rangle = \langle w | P\pi(b)(v) \rangle. \end{aligned}$$

So $P \in \pi(\mathcal{B})'$. Let $\rho(a)$ be $\langle h | \varphi(a)h \rangle$ ($a \in \mathcal{A}$). Then, for any $b \in \mathcal{B}$,

$$\begin{aligned} \rho(b) &= \langle h | \pi(b)h \rangle = \langle h | P\pi(b)h \rangle = \langle P(h) | \pi(b)P(h) \rangle = \langle y | \pi(b)y \rangle \\ &= \langle y | \pi_{\mathcal{B}}(b)y \rangle = \langle \sqrt{t}x | \pi_{\mathcal{B}}(b)(\sqrt{t}x) \rangle = t\rho'(b), \end{aligned}$$

which implies that $\rho|_{\mathcal{B}} = t\rho'$. Besides, it is obvious that ρ belongs to $\mathcal{P}_{\mathcal{A},[\pi]} \cap \mathcal{P}^{\mathcal{B}}(\mathcal{A})$. Based on the discussion above, we know that ρ belongs to pre-image $\Theta^{-1}(t, \rho')$. So (b) holds. \square

Remark 2.8. *If $\pi \in \hat{\mathcal{A}}^{\mathcal{B}}$ and $\Delta := \Theta^{-1}(\{1\} \times \mathcal{P}_{\mathcal{B},[\pi_{\mathcal{B}}]})$, then $\Theta|_{\Delta}$ can be identified with the inverse map of Ξ .*

Moreover, we can describe the relation between $\Xi(\mathcal{P}(\mathcal{B}))$ and $\mathcal{P}^{\mathcal{B}}(\mathcal{A})$ by endowing a kind of geometric structure on their pure state spaces.

For any $\mu \in \mathcal{P}(\mathcal{A})$ and $t \in (0, +\infty)$, set $\mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, $S(\mu; t) := \{\nu \in \mathcal{P}(\mathcal{A}) \mid d_{\mathcal{A}}(\mu, \nu) = t\}$, $D(\mu; t) := \{\nu \in \mathcal{P}(\mathcal{A}) \mid d_{\mathcal{A}}(\mu, \nu) < t\}$ and $B(\mu; t) := \{\nu \in \mathcal{P}(\mathcal{A}) \mid d_{\mathcal{A}}(\mu, \nu) \leq t\}$.

Theorem 2.9. *Let \mathcal{B} be a nonzero hereditary C^* -subalgebra of a C^* -algebra \mathcal{A} , and set $\kappa := \frac{\sqrt{2}\pi}{2}$. For any $[\pi] \in \hat{\mathcal{A}}^{\mathcal{B}}$, $\mu \in \mathcal{P}_{\mathcal{B}, [\pi]_{\mathcal{B}}}$ and $t \in (0, \kappa)$, we have*

(a)

$$p_{\mathcal{A}}^{-1}(\hat{\mathcal{A}} \setminus \hat{\mathcal{A}}^{\mathcal{B}}) = \{\rho \in \mathcal{P}(\mathcal{A}) \mid d_{\mathcal{A}}(\rho, \Xi(\mathcal{P}(\mathcal{B}))) = 3\},$$

$$\mathcal{P}^{\mathcal{B}}(\mathcal{A}) = \{\rho \in \mathcal{P}(\mathcal{A}) \mid d_{\mathcal{A}}(\rho, \Xi(\mathcal{P}(\mathcal{B}))) < \kappa\},$$

and

$$p_{\mathcal{A}}^{-1}(\hat{\mathcal{A}}^{\mathcal{B}}) \setminus \mathcal{P}^{\mathcal{B}}(\mathcal{A}) = \{\rho \in \mathcal{P}(\mathcal{A}) \mid d_{\mathcal{A}}(\rho, \Xi(\mathcal{P}(\mathcal{B}))) = \kappa\}.$$

(b) $\mathcal{P}_{\mathcal{A}, [\pi]} = B(\Xi(\mu); \kappa)$, $\mathcal{P}_{\mathcal{A}, [\pi]} \cap \mathcal{P}^{\mathcal{B}}(\mathcal{A}) = D(\Xi(\mu); \kappa)$, and $\mathcal{P}_{\mathcal{A}, [\pi]} \setminus \mathcal{P}^{\mathcal{B}}(\mathcal{A}) = S(\Xi(\mu); \kappa)$.

(c) $\Theta^{-1}(\cos^2(\frac{t}{\sqrt{2}}), \mu) = S(\Xi(\mu); t)$ and there is a bijection

$$\Upsilon : S(\Xi(\mu); t) \rightarrow \mathbb{T} \times (\mathcal{P}_{\mathcal{A}, [\pi]} \setminus \mathcal{P}^{\mathcal{B}}(\mathcal{A})).$$

Proof: According to Proposition 2.7, we obtain two surjections

$$\hat{\Theta} : \mathcal{P}(\mathcal{A})^{\mathcal{B}} \rightarrow \mathcal{P}(\mathcal{B}), \rho \mapsto \rho',$$

and

$$t_{\mathcal{B}} : \mathcal{P}(\mathcal{A})^{\mathcal{B}} \rightarrow (0, 1], \rho \mapsto t,$$

where $(t, \rho') = \Theta(\rho)$. It is apparent that $t_{\mathcal{B}}$ can be extended to $\mathcal{P}(\mathcal{A})$ by setting

$$t_{\mathcal{B}}(\rho) = 0, \forall \rho \in \mathcal{P}(\mathcal{A}) \setminus \mathcal{P}(\mathcal{A})^{\mathcal{B}}.$$

For any $\rho \in \mathcal{P}(\mathcal{A})^{\mathcal{B}}$, let $(\mathcal{H}_{\rho}, \pi_{\rho}, x_{\rho})$ be the GNS representation of \mathcal{A} associated to ρ . One may identify $\mathcal{H}_{\hat{\Theta}(\rho)} = \pi_{\rho}(\mathcal{B})\mathcal{H}_{\rho} \subseteq \mathcal{H}_{\rho}$ and $\pi_{\hat{\Theta}(\rho)}(b) = \pi_{\rho}(b)|_{\pi_{\rho}(\mathcal{B})\mathcal{H}_{\rho}}$. Moreover, there is a unique $w_{\rho} \in \mathfrak{S}_1(\mathcal{H}_{\rho} \ominus \mathcal{H}_{\hat{\Theta}(\rho)})$ such that

$$x_{\rho} = \sqrt{t_{\mathcal{B}}(\rho)}x_{\hat{\Theta}(\rho)} + \sqrt{1 - t_{\mathcal{B}}(\rho)}w_{\rho},$$

which implies that

$$d_{\mathcal{A}}(\rho, \Xi(\hat{\Theta}(\rho))) = \sqrt{2} \arccos \sqrt{t_{\mathcal{B}}(\rho)}.$$

Consequently,

$$\mathcal{P}(\mathcal{A})^{\mathcal{B}} = \bigcup_{\omega \in \mathcal{P}(\mathcal{B})} D(\Xi(\omega); \kappa) \subseteq p_{\mathcal{A}}^{-1}(\hat{\mathcal{A}}^{\mathcal{B}}) = \bigcup_{\omega \in \mathcal{P}(\mathcal{B})} B(\Xi(\omega); \kappa).$$

From the above argument, it can be easily checked that (a) holds.

For any $\mu \in \mathcal{P}_{\mathcal{B}, [\pi]_{\mathcal{B}}}$ and $\rho \in \mathcal{P}_{\mathcal{A}, [\pi]}$, one may identify $\mathcal{H}_{\rho} = \mathcal{H}_{\pi}$ and $\mathcal{H}_{\mu} = \mathcal{H}_{\pi_{\mathcal{B}}}$. So there is a unique $w_{\rho} \in \mathfrak{S}_1(\mathcal{H}_{\pi} \ominus \mathcal{H}_{\pi_{\mathcal{B}}})$ such that $x_{\rho} = \sqrt{t_{\mathcal{B}}(\rho)}x_{\mu} + \sqrt{1 - t_{\mathcal{B}}(\rho)}w_{\rho}$. For any $t \in (0, \kappa)$, we have $\Theta^{-1}(\cos^2(\frac{t}{\sqrt{2}}), \mu)$ is the subset of $\mathcal{P}_{\mathcal{A}, [\pi]}$, which is as follow:

$$\{\rho \in \mathcal{P}_{\mathcal{A}, [\pi]} \mid x_{\rho} = \cos(\frac{t}{\sqrt{2}})x_{\mu} + \sqrt{1 - \cos^2(\frac{t}{\sqrt{2}})}w, \forall w \in \mathfrak{S}_1(\mathcal{H}_{\pi} \ominus \mathcal{H}_{\pi_{\mathcal{B}}})\},$$

which implies that

$$\Theta^{-1}(\cos^2(\frac{t}{\sqrt{2}}), \mu) \cong \mathfrak{S}_1(\mathcal{H}_\pi \ominus \mathcal{H}_{\pi_{\mathcal{B}}}) \cong \mathbb{T} \times \mathcal{P}_{\mathcal{H}_\pi \ominus \mathcal{H}_{\pi_{\mathcal{B}}}}.$$

In addition, we have that

$$d_{\mathcal{A}}(\rho, \Xi(\mu)) = \sqrt{2} \arccos \sqrt{t_{\mathcal{B}}(\rho)}, \Phi_{\mathcal{A}, [\pi]}(\mathcal{P}_{\mathcal{A}, [\pi]} \setminus \mathcal{P}^{\mathcal{B}}(\mathcal{A})) = \mathcal{P}_{\mathcal{H}_\pi \ominus \mathcal{H}_{\pi_{\mathcal{B}}}}.$$

So (b) and (c) hold. \square

Next, we characterize the relationship between $\mathcal{P}(\mathcal{B})$ and $\mathcal{P}(\mathcal{A})$ by using the notion of Kähler bundle. Before that, we recall the following well-known result.

Lemma 2.10. *Let \mathcal{I} be a closed ideal which is generated by a nonzero hereditary C^* -subalgebra \mathcal{B} of a C^* -algebra \mathcal{A} . Then $\hat{\mathcal{A}}^{\mathcal{B}} = \hat{\mathcal{A}}^{\mathcal{I}}$ is an open set in $\hat{\mathcal{A}}$. Moreover, one has $\mathcal{I} = \bigcap_{\mathcal{J} \in \text{hull}(\mathcal{B})} \mathcal{J}$.*

Definition 2.11. ([8, Definition 8.5]) *A closed (or open) subset \mathcal{N} of a Kähler Hilbert manifold \mathcal{M} is called a closed (or open) submanifold if for any $x \in \mathcal{N}$, there exists a chart $(\mathcal{V}, b_{\mathcal{V}}, \mathcal{H}_{\mathcal{V}})$ of x and a closed subspace \mathcal{E} of $\mathcal{H}_{\mathcal{V}}$ such that $b_{\mathcal{V}}(\mathcal{N} \cap \mathcal{V}) = \mathcal{E} \cap b_{\mathcal{V}}(\mathcal{V})$.*

Lemma 2.12. *If \mathcal{H} is a Hilbert space and \mathcal{M} is a closed subspace of \mathcal{H} , then $\mathcal{P}_{\mathcal{M}}$ is a closed Kähler submanifold of $\mathcal{P}_{\mathcal{H}}$.*

Proof: For any $[\xi] \in \mathcal{P}_{\mathcal{M}}$, let $(\mathcal{V}_{\xi}, b_{\xi}, \mathcal{H}_{\xi})$ be a canonical chart of $[\xi]$ (see Appendix C in [1]). Since $b_{\xi}(\mathcal{P}_{\mathcal{M}} \cap \mathcal{V}_{\xi}) = \mathcal{M} \cap \mathcal{H}_{\xi}$, it follows that $\mathcal{P}_{\mathcal{M}}$ is a Kähler submanifold of $\mathcal{P}_{\mathcal{H}}$. Moreover, it can be easily checked that $\mathcal{P}_{\mathcal{M}}$ is closed under the Kähler topology induced by the Kähler distance $d_{\mathcal{H}}$. \square

Theorem 2.13. *Let \mathcal{B} be a nonzero hereditary C^* -subalgebra of a C^* -algebra \mathcal{A} . Then*

(a) *$(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})^{\mathcal{B}} := (\Xi(\mathcal{P}(\mathcal{B})), p_{\mathcal{A}}, \hat{\mathcal{A}}^{\mathcal{B}})$ is a Kähler subbundle of $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})$ such that $\Xi(\mathcal{P}_{\mathcal{B}, [\pi_{\mathcal{B}}]})$ is a closed Kähler submanifold of $\mathcal{P}_{\mathcal{A}, [\pi]}$ for all $[\pi] \in \hat{\mathcal{A}}^{\mathcal{B}}$. Moreover, $(\mathcal{P}(\mathcal{B}), p_{\mathcal{B}}, \hat{\mathcal{B}})$ is uniformly Kähler isomorphic to $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})^{\mathcal{B}}$.*

(b) *\mathcal{B} is a closed ideal if and only if $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})^{\mathcal{B}} = (\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})_{\hat{\mathcal{A}}^{\mathcal{B}}}$.*

Proof: (a) We denote by Ψ the canonical homeomorphism from $\hat{\mathcal{A}}^{\mathcal{B}}$ to $\hat{\mathcal{B}}$. Let Ξ be a homeomorphism from $\mathcal{P}(\mathcal{B})$ to $\Xi(\mathcal{P}(\mathcal{A}))$ satisfying $\Psi^{-1} \circ p_{\mathcal{B}} = p_{\mathcal{A}} \circ \Xi$. By Proposition 2.7, for any $[\pi] \in \hat{\mathcal{A}}^{\mathcal{B}}$, we have $\Xi(\mathcal{P}_{\mathcal{B}, [\pi_{\mathcal{B}}]}) \subseteq \mathcal{P}_{\mathcal{A}, [\pi]}$. So $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})^{\mathcal{B}}$ is a subbundle of $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})$. Since $\Phi_{\mathcal{A}, [\pi]}(\mathcal{P}_{\mathcal{A}, [\pi]}) = \mathcal{P}_{\mathcal{H}_\pi}$ and $\Phi_{\mathcal{B}, [\pi_{\mathcal{B}}]}(\mathcal{P}_{\mathcal{B}, [\pi_{\mathcal{B}}]}) = \mathcal{P}_{\mathcal{H}_{\pi_{\mathcal{B}}}}$, according to Lemma 2.12, we know that $\Xi(\mathcal{P}_{\mathcal{B}, [\pi_{\mathcal{B}}]})$ is a closed Kähler submanifold of $\mathcal{P}_{\mathcal{A}, [\pi]}$. Moreover, according to the above argument, it is obvious that (Ξ, Φ^{-1}) is a uniform Kähler isomorphism between $(\mathcal{P}(\mathcal{B}), p_{\mathcal{B}}, \hat{\mathcal{B}})$ and $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})^{\mathcal{B}}$. Hence (a) holds.

(b) It is obvious by Proposition 2.5(a) and Lemma 2.12. \square

Finally, we give a concrete example: \mathcal{A} is $\mathcal{K}(\mathcal{H})$ consisting of all compact operators on a Hilbert space \mathcal{H} .

It can be easily proved that the UKB associated with \mathcal{A} has only one fibre denoted by \mathcal{P} because $\hat{\mathcal{A}} = \{[(\mathcal{H}, i)]\}$ (see [6, Example 5.1.1]), where i is the inclusion map from \mathcal{A} to $\mathcal{L}(\mathcal{H})$. We denote by Φ the canonical Kähler isomorphism from \mathcal{P} to $\mathcal{P}_{\mathcal{H}}$ and by π the quotient map from \mathcal{H} to $\mathcal{P}_{\mathcal{H}}$. For any $[\xi] \in \mathcal{P}_{\mathcal{H}}$, set $\{\xi\}^\perp := \{x \in \mathcal{H} \mid \langle \xi | x \rangle = 0\}$ and denote by $[S]$ the closed linear span of a set $S \subset \mathcal{H}$.

Note that $\{\xi\}^\perp \cong T_{[\xi]}\Phi(\mathcal{P})$, where $T_{[\xi]}\Phi(\mathcal{P})$ is the *holomorphic tangent space* (See Appendix A in [1] and Section 4 in [8]) at $[\xi]$ in the Kähler manifold $\Phi(\mathcal{P})$.

Proposition 2.14. *Let \mathcal{A} be $\mathcal{K}(\mathcal{H})$. Then $\mathcal{B} \subseteq \mathcal{A}$ is a hereditary C^* -subalgebra if and only if the UKB associated with \mathcal{B} has only one fibre \mathcal{P}' which is a closed Kähler submanifold of \mathcal{P} satisfying*

$$\pi([\sum_{x \in \mathcal{P}'} T_{\Phi(x)}\Phi(\mathcal{P}')]) \subset \Phi(\mathcal{P}').$$

Moreover, one has $\mathcal{P}' = \Phi^{-1}(\mathcal{P}_{\mathcal{M}})$ and $\mathcal{B} = \mathcal{K}(\mathcal{M})$, where $\mathcal{M} := [\sum_{x \in \mathcal{P}'} T_{\Phi(x)}\Phi(\mathcal{P}')]$.

Proof: It is well known that \mathcal{B} is a hereditary C^* -subalgebra of $\mathcal{K}(\mathcal{H})$ if and only if there is a closed subspace \mathcal{M} of \mathcal{H} such that $\mathcal{B} = \mathcal{K}(\mathcal{M})$. So the forward implication is obvious considering $\hat{\mathcal{A}} = \{[(\mathcal{H}, i)]\}$.

Conversely, assume that \mathcal{P}' is a closed Kähler submanifold of \mathcal{P} such that

$$\pi([\sum_{x \in \mathcal{P}'} T_{\Phi(x)}\Phi(\mathcal{P}')]) \subset \Phi(\mathcal{P}').$$

Choose an arbitrary point $[\xi]$ in $\Phi(\mathcal{P}') \subset \Phi(\mathcal{P})$, and let $(\mathcal{V}_\xi, b_\xi, \mathcal{H}_\xi)$ be a chart of $[\xi]$. Since \mathcal{P}' is a closed Kähler submanifold of \mathcal{P} , there is a chart $(\mathcal{V}_\xi \cap \Phi(\mathcal{P}'), b_\xi|_{\mathcal{V}_\xi \cap \Phi(\mathcal{P}')}, \mathcal{W}_\xi)$ of $\Phi(\mathcal{P}')$ such that $b_\xi(\mathcal{V}_\xi \cap \Phi(\mathcal{P}')) = \mathcal{W}_\xi \cap b_\xi(\mathcal{V}_\xi)$. By the assumption, we have

$$\pi(T_{[\eta]}\Phi(\mathcal{P}')) \subset \Phi(\mathcal{P}'), \forall [\eta] \in \Phi(\mathcal{P}').$$

It follows from the definition of b_ξ that

$$\pi(\{\eta\} \oplus T_{[\eta]}\Phi(\mathcal{P}')) \subset \Phi(\mathcal{P}'), \forall [\eta] \in \Phi(\mathcal{P}').$$

For each $[\zeta] \in \Phi(\mathcal{P}')$, there exists $[\eta] \in \Phi(\mathcal{P}')$ such that $\zeta \in T_{[\eta]}(\Phi(\mathcal{P}'))$. Hence

$$\Phi(\mathcal{P}') \subset \bigcup_{[\eta] \in \Phi(\mathcal{P}')} \pi(T_{[\eta]}\Phi(\mathcal{P}')).$$

Since

$$\bigcup_{[\eta] \in \Phi(\mathcal{P}')} \pi(T_{[\eta]}\Phi(\mathcal{P}')) \subset \pi([\sum_{x \in \mathcal{P}'} T_{\Phi(x)}\Phi(\mathcal{P}')]),$$

we have

$$\Phi(\mathcal{P}') \subset \pi([\sum_{x \in \mathcal{P}'} T_{\Phi(x)}\Phi(\mathcal{P}')]).$$

Consequently,

$$\Phi(\mathcal{P}') = \pi([\sum_{x \in \mathcal{P}'} T_{\Phi(x)}\Phi(\mathcal{P}')]).$$

Thus, we obtain that $\mathcal{P}' = \Phi^{-1}(\mathcal{P}_{\mathcal{M}})$ and $\mathcal{B} = \mathcal{K}(\mathcal{M})$, where $\mathcal{M} := [\sum_{x \in \mathcal{P}'} T_{\Phi(x)}\Phi(\mathcal{P}')]$. \square

Remark 2.15. Let \mathcal{A} be a C^* -algebra. Consider \mathcal{A} as a right Hilbert module over itself, i.e., with the \mathcal{A} -valued inner product $(a, b) \mapsto a^*b$ ($\forall a, b \in \mathcal{A}$). According to [3], \mathcal{A} can be view as a uniform holomorphic Hilbert bundle $\{\mathcal{H}(\mathcal{A})_\rho\}_{\rho \in \mathcal{P}(\mathcal{A}) \cup \{0\}}$, where $\mathcal{H}(\mathcal{A})_\rho$ is the GNS Hilbert space associated to ρ . Hence, each C^* -algebra \mathcal{A} can induce two bundles $\{\mathcal{H}(\mathcal{A})_\rho\}_{\rho \in \mathcal{P}(\mathcal{A}) \cup \{0\}}$ and $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})$ (the base space of the former bundle is the total space of the latter bundle).

It is well known that the correspondence $\mathcal{L} \mapsto \mathcal{L} \cap \mathcal{L}^*$ is a bijection from the set of closed left ideals of \mathcal{A} onto the set of hereditary C^* -subalgebras of \mathcal{A} . We denote by $\mathcal{L}(\mathcal{B})$ the closed left ideal associated to a hereditary C^* -subalgebra \mathcal{B} of \mathcal{A} . Since \mathcal{L}^* is a closed right ideal in \mathcal{A} , \mathcal{L}^* can be considered as a right Hilbert sub- \mathcal{A} -module of \mathcal{A} . So it follows from [3] that \mathcal{L} can induce a subbundle $\{\mathcal{H}(\mathcal{L}^*)_\rho\}_{\rho \in \mathcal{P}(\mathcal{A}) \cup \{0\}}$ of $\{\mathcal{H}(\mathcal{A})_\rho\}_{\rho \in \mathcal{P}(\mathcal{A}) \cup \{0\}}$. Consequently, \mathcal{B} corresponds to such subbundle.

In brief, each hereditary C^* -subalgebra \mathcal{B} of \mathcal{A} can be described as two different bundles associated to \mathcal{A} respectively, i.e., a uniform Kähler subbundle $(\mathcal{P}(\mathcal{A}), p_{\mathcal{A}}, \hat{\mathcal{A}})^\mathcal{B}$ and a uniform holomorphic Hilbert subbundle $\{\mathcal{H}(\mathcal{L}(\mathcal{B})^*)_\rho\}_{\rho \in \mathcal{P}(\mathcal{A}) \cup \{0\}}$.

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